On some solutions of the Dirac equation

1996 J. Phys. A: Math. Gen. 29 2827

(http://iopscience.iop.org/0305-4470/29/11/016)

J. Phys. A: Math. Gen. **29** (1996) 2827–2832. Printed in the UK

On some solutions of the Dirac equation

G Lévai† and A Del Sol Mesa‡§

† Institute of Nuclear Research of the Hungarian Academy of Sciences, Debrecen, Pf 51, Hungary 4001

‡ Sloane Physics Laboratory, Yale University, PO Box 6666, New Haven, CT 09511, USA

Received 21 November 1995

Abstract. The solutions of the Dirac equation with minimal and non-minimal coupling terms are investigated by transforming the relativistic equation into a Schrödinger-like one. Earlier results are discussed in a unified framework and certain solutions of a large class of potentials are given. It is pointed out that techniques used in the analysis of quasi-exactly solvable potentials of non-relativistic quantum mechanics can be applied to relativistic problems as well.

The introduction of exactly solvable potentials in the Dirac equation has been a subject of much discussion [1–4]. The electromagnetic potentials can be considered according to the minimal substitution $p^\mu \to p^\mu - (e/c)A^\mu$, in this equation, where A^μ is the 4-vector potential.

One of the examples of the Dirac equation with a minimal substitution, which can be solved exactly, is of course the relativistic hydrogen atom [5] corresponding to $A_i = 0, i = 1, 2, 3$ and $(e/c)A_0 \equiv \phi$, where $\phi = -(e^2/r)$.

A non-minimal substitution in the Dirac equation gives other kinds of interactions. Among these problems we have, for example, the so-called Dirac oscillator named by Moshinsky and Szczepaniak [1] and studied before by Cook [2]. The idea of the Dirac oscillator is the non-minimal substitution $\boldsymbol{p} \to \boldsymbol{p} - \mathrm{i}\omega m \boldsymbol{r}\beta$, where ω is the frequency of the oscillator, m the mass of the particle, $\boldsymbol{r}$ the position vector and $\beta = \gamma^0$.

The interest in the paper of Moshinsky and Szczepaniak has given rise to a number of investigations concerning its covariance [6], its symmetry properties [7] and its generalization to many-particle systems [8]. Also, there have been some works connected with the search of new interactions in the Dirac equation [3, 4]. Among the papers on this matter we would like to cite the one of Castaños *et al* [3], where the authors propose a large class of Dirac oscillator-type couplings, paying attention to the supersymmetry properties of these systems. On the other hand, in the paper of Domínguez-Adame and González [4] a particular example of minimal and non-minimal coupling is studied.

Our present work was motivated by the idea of connecting the methods used in the analysis of exactly solvable potentials in non-relativistic quantum mechanics with the solution procedure of the Dirac equation containing various interaction terms. To this end we apply an inverse method: we start with a general expression for the minimal and non-minimal couplings in the Dirac equation, and then we reduce this equation to its radial form, in order to study some families of potentials which could be solved exactly or quasi-exactly, making a similar analysis as in the search of solvable or quasi-exactly solvable potentials in non-relativistic quantum mechanics.

§ Permanent address: Instituto de Física, UNAM, Apdo Postal 20-264, 01000 México DF, Mexico.

0305-4470/96/112827+06$19.50

In fact, for some specific values of our parameters, we cover some examples analysed by other authors [3, 4], and discuss some polynomial potentials, such as the sextic anharmonic oscillator, which is an example of a quasi-exactly solvable potential.

We consider the Dirac equation of the form ($c = \hbar = 1$)

$$\left[\boldsymbol{\alpha}\cdot(\boldsymbol{p} - \mathrm{i}\beta v(r)\boldsymbol{r} - u(r)\boldsymbol{r}) + m\beta - E\right]\Psi = 0 \tag{1}$$

where $v(r)$ and $u(r)$ are some functions of r and $\boldsymbol{\alpha}$ and β are defined as

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2}$$

Writing down equation (1) separately for the two components Ψ_1 and Ψ_2 of

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

we get

$$(m - E)\Psi_1 + \boldsymbol{\sigma}\cdot\left[\boldsymbol{p} + (\mathrm{i}v(r) - u(r))\boldsymbol{r}\right]\Psi_2 = 0 \tag{3a}$$

$$\boldsymbol{\sigma}\cdot\left[\boldsymbol{p} - (\mathrm{i}v(r) + u(r))\boldsymbol{r}\right]\Psi_1 - (m + E)\Psi_2 = 0. \tag{3b}$$

Applying the standard procedure this reduces to the following equation for the Ψ_1 component:

$$\left[\boldsymbol{p}^2 - 2u(r)\boldsymbol{r}\cdot\boldsymbol{p} + (v^2(r) + u^2(r))\boldsymbol{r}^2 - 2v(r)\boldsymbol{\sigma}\cdot\boldsymbol{L} - r\left(\frac{\mathrm{d}v}{\mathrm{d}r} - \mathrm{i}\frac{\mathrm{d}u}{\mathrm{d}r}\right) - 3(v(r) - \mathrm{i}u(r))\right]\Psi_1$$
$$= (E^2 - m^2)\Psi_1. \tag{4}$$

Separating the radial, angular and spin variables by writing Ψ_1 as

$$\Psi_1 = f(r)\frac{1}{r}\left|\left(l\frac{1}{2}\right)jm_j\right\rangle \tag{5}$$

the following radial equation is obtained for $f(r)$:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + 2\mathrm{i}ru(r)\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l(l+1)}{r^2} + (rv(r))^2 - \frac{\mathrm{d}}{\mathrm{d}r}(rv(r)) - 2(K+1)v(r)\right.$$
$$\left. + (ru(r))^2 + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r}(ru(r)) - \epsilon\right]f(r) = 0 \tag{6}$$

where

$$K = j(j+1) - l(l+1) - \tfrac{3}{4} = \begin{cases} l = j - \frac{1}{2} & \text{if } j = l + \frac{1}{2} \\ -l - 1 = -j - \frac{3}{2} & \text{if } j = l - \frac{1}{2} \end{cases} \tag{7a}$$

and

$$\epsilon = E^2 - m^2. \tag{7b}$$

Considering the following functional form for $f(r)$:

$$f(r) = r^{\nu} \exp(-z(r))\Phi(r) \tag{8}$$

we find after straightforward calculation that this $f(r)$ solves equation (6) for the special case of $\Phi(r) =$ constant and $\epsilon = 0$ (i.e. $E^2 = m^2$) if $z(r)$ is chosen as

$$z(r) = \int^{r} r'(-\mathrm{i}u(r') + v(r'))\,\mathrm{d}r' \tag{9}$$

provided that

$$\nu(\nu - 1) = l(l + 1) \tag{10a}$$

and

$$\nu = K + 1 \tag{10b}$$

hold. The solutions of (10*a*) $\nu = l + 1$ and $-l$, automatically satisfy (10*b*) for the $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ cases, respectively. Up to this point the functions $v(r)$ and $u(r)$ have not been specified yet.

Further solutions of (6) for $\epsilon \neq 0$ can be obtained by using $\Phi(r) \neq$ constant, which leads to a second-order differential equation for $\Phi(r)$. From the possible choices for $\Phi(r)$ in what follows we consider the functional form

$$\Phi(r) = F(\sigma, \rho; g(r)) \tag{11}$$

where $F(\sigma, \rho; g)$ solves the confluent hypergeometric differential equation

$$g\frac{\mathrm{d}^2 F}{\mathrm{d}g^2} + (\rho - g)\frac{\mathrm{d}F}{\mathrm{d}g} - \sigma F = 0. \tag{12}$$

When transforming equation (6) into (12) by the use of $f(r)$ in (8), (9) and (11) we arrive at the following expressions for the previously unspecified functions $g(r)$ and $v(r)$:

$$g(r) = a\ r^2 \qquad (a > 0) \tag{13a}$$

$$v(r) = a + \frac{b}{r^2} \tag{13b}$$

and also get

$$\rho = \nu + \tfrac{1}{2} + b \tag{14a}$$

and

$$\sigma = -\frac{\epsilon}{4a} - \frac{K + 1}{2} + \frac{\nu}{2} \tag{14b}$$

with

$$\nu = \begin{cases} b + \frac{1}{2} + |j - b| & \text{for } j = l + \frac{1}{2} \\ b + \frac{1}{2} + |j + 1 + b| & \text{for } j = l - \frac{1}{2}. \end{cases} \tag{14c}$$

Note, however, that we have not obtained any restictions for $u(r)$ yet.

The wavefunctions take the form

$$f^{(+)}(r) = r^{\frac{1}{2}+|j-b|} \exp(-\tfrac{1}{2}ar^2 + \mathrm{i}w(r)) F(-n_r, 2b + 1 + |j - b|; ar^2) \tag{15a}$$

and

$$f^{(-)}(r) = r^{\frac{1}{2}+|j+1+b|} \exp(-\tfrac{1}{2}ar^2 + \mathrm{i}w(r)) F(-n_r, 2b + 1 + |j + 1 + b|; ar^2) \tag{15b}$$

where we have used superscripts (+) and (−) to distinguish between the cases for $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$, respectively, and $w(r)$ is defined as

$$w(r) = \int^r r' u(r')\, \mathrm{d}r'. \tag{16}$$

This means that the function $u(r)$ which has not been specified up to this point contributes to a phase factor. The corresponding energy eigenvalues are obtained from

$$\epsilon^{(+)} \equiv (E^{(+)})^2 - m^2 = 2a(N - 2j + \tfrac{1}{2} + b + |j - b|) \tag{17a}$$

and

$$\epsilon^{(-)} \equiv (E^{(-)})^2 - m^2 = 2a(N + \tfrac{1}{2} + b + |j + 1 + b|) \tag{17b}$$

with $N = 2n_r + l$.

These results can be interpreted as the generalization of the Dirac oscillator [1], which corresponds to $a = m\omega$, $b = 0$ and $u(r) = 0$. The extension of the Dirac oscillator by Castaños *et al* [3] is also included in these formulae with $u(r) = 0$, although the energy eigenvalues published in that work differ slightly from those in (17) due to a different parametrization used by the authors. In addition to these results, equations (16) and (17) also include another extension of the Dirac oscillator by Domínguez-Adame and González [4] as a special case, where a linear potential has been considered in the minimal coupling term in addition to the Dirac oscillator. This situation corresponds to taking $a = m\omega_s$, $b = 0$ and $u(r) = \mathrm{i}m\omega_v$. These authors also noted that the appearance of ω_v does not modify the energy spectrum, and the new term with respect to the Dirac oscillator influences only the form of the wavefunctions. A simple explanation for this result can be given remembering that $u(r)$ basically represented a phase factor. This is not evident from the formulae presented in [4]; nevertheless, one should remember that chosing an imaginary, rather than a real, $u(r)$ would break the Hermiticity of the Hamiltonian in (1).

We note that similar extensions of other solvable Dirac equations also seem possible by considering functional forms of $f(r)$ other than that in (8), (9) and (11). In order to accomodate the Coulomb problem in this procedure one has to allow state-dependent (i.e. quantum-number-dependent) functional forms for $g(r)$ and $z(r)$, as a result of which $f(r)$ ceases to be separable into the product of the ground-state wavefunction and a confluent hypergeometric function (see, e.g., [9]). Our procedure could be applied to the Coulomb problem by incorporating the above modifications, but this is beyond the scope of this work. We only mention that substituting $v(r) = ar^{-1} + br^{-2}$ and $u(r) = 0$ in (6) leads to the radial Coulomb Schrödinger equation, which then can be solved by standard procedures. (This is the second example of solvable Dirac equations presented by Castaños *et al* [3].)

Another possible way is transforming the second-order differential equation in (6) into the differential equation of other special functions. However, solvable problems associated with the hypergeometric function (i.e. the Natanzon potentials [10] in non-relativistic quantum mechanics, for example) have the drawback of being solvable for radial problems with $l = 0$ only, which restricts their applicability in areas discussed here. The structure of the Bessel equation also forbids the application of Bessel functions as $\Phi(r)$ in (8).

Other solvable problems allowing solutions for orbital angular momenta l other than zero include, for example, polynomial potentials. These are usually interpreted as anharmonic oscillators and are examples of quasi-exactly-solvable potentials [11]. These problems cannot be solved in general, except for some special values of the potential parameters when a finite number of exact energy eigenvalues can be determined together with the corresponding wavefunctions. Such potentials can easily be accomodated in our approach by substituting polynomial forms of $v(r)$ in (6). The solutions can then be written as $f(r)$ in (8) and (9) with $\Phi(r)$ being a polynomial. The solution for the ground state (i.e. $\epsilon = 0$) can immediately be given by setting $\Phi(r) =$ constant. The sextic anharmonic oscillator [12], for example, corresponds to choosing

$$v(r) = c_2 r^2 + c_0 \qquad u(r) = 0 \tag{18}$$

and equation (6) reduces to

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + c_2^2 r^6 + 2c_2 c_0 r^4 + (c_0^2 - c_2(2K+5))r^2 - (\epsilon + c_0(2K+3))\right] f(r) = 0 \tag{19}$$

in this case, with K defined in (7*a*). Note that the coefficient of the sextic term is always positive, which guarantees confinement. Also note that the quadratic term carries j-dependence.

The ground-state solution for each $j = l + \frac{1}{2}$ is readily determined by substituting $v(r)$ in (18) into equations (9) and (8) with (10). This solvability is the result of the particular structure of (6), which leads to a correlated behaviour of the potential parameters (mentioned above) characterizing quasi-exactly solvable potentials. Other polynomial potentials containing odd powers of r (like the Coulomb term) as well can also be solved directly for $\epsilon^{(+)} = 0$ and $j = l + \frac{1}{2}$ by this method. In the $j = l - \frac{1}{2}$ case, however, only the solutions singular at the origin ($\simeq r^{-l}$) can be obtained, as can be seen from (8) and (10). Similar to the extension of the Dirac oscillator in the spirit of (19) the Dirac–Coulomb problem can also be supplemented with additional potential terms.

It is remarkable that equations (8), (9) and (10) solve the differential equation (6) for $E^2 = m^2$ and $j = l + \frac{1}{2}$ (with $\Phi(r) =$ constant) without any further assumptions. Besides $v(r)$ functions in a quasi-exactly solvable type equation in (6) this applies to non-quasi-exactly solvable problems and even to non-polynomial forms of $v(r)$. The solutions obtained this way are nodeless ones and can readily be written down for any l. In the particular case of the sextic oscillator (19) we find that $\epsilon^{(+)} = (E^{(+)})^2 - m^2$ represents a solution for any l in the $j = l + \frac{1}{2}$ case, which means that the infinite degeneracy of the Dirac oscillator [1] is maintained for the $(E^{(+)})^2 = m^2$ levels if sextic (and correlated quadratic) terms are introduced. An interesting task would be to investigate whether the extra terms lift the infinite degeneracy of the other levels and also the finite degeneracy of states with $j = l - \frac{1}{2}$. The solution of these problems, however, seems to require more complex functional forms than that in (8) with a polynomial $\Phi(r)$. This is because the Hill determinant method [13] fails to supply physically acceptable solutions other than that with $E^2 = m^2$ and $j = l + \frac{1}{2}$ if $\Phi(r)$ is chosen to be a polynomial. Further analysis of this and similar problems would be worthwhile.

In conclusion, we have studied the solutions of the Dirac equation by applying techniques used in the analysis of the Schrödinger equation. We have identified several known exactly solvable relativistic problems as special cases of our approach and extended the range of (partially) solvable Dirac equations by pointing out the relevance of quasi-exactly solvable potentials to relativistic problems.

Acknowledgments

The authors thank Professor F Iachello for his hospitality at Yale University. This work was also supported by grant JF 345/93 of the US–Hungarian Science and Technology Joint Fund and by grant F4303 of the OTKA (Hungary).

References

[1] Moshinsky M and Szczepaniak A 1989 *J. Phys. A: Math. Gen.* **22** L817
[2] Cook P A 1971 *Lett. Nuovo Cimento* **1** 419
[3] Castaños O, Frank A, López R and Urrutia L F 1991 *Phys. Rev.* D **43** 544
[4] Domínguez-Adame F and González M A 1990 *Europhys. Lett.* **13** 193
[5] Landau L D and Lifshitz E M 1977 *Quantum Mechanics* (Oxford: Pergamon)
[6] Moreno M and Zentella A 1989 *J. Phys. A: Math. Gen.* **22** L821
[7] Quesne C and Moshinsky M 1990 *J. Phys. A: Math. Gen.* **23** 2263
[8] Del Sol Mesa A and Moshinsky M. 1994 *J. Phys. A: Math. Gen.* **27** 4685
Moshinsky M and Del Sol Mesa A 1994 *Can. J. Phys.* **27** 453
[9] Lévai G 1989 *J. Phys. A: Math. Gen.* **22** 689
[10] Natanzon G A 1979 *Teor. Mat. Fiz.* **38** 146
[11] Ushveridze A K 1994 *Quasi-exactly solvable Models in Quantum Mechanics* (Bristol: Institute of Physics)
[12] Singh V, Biswas S N and Datta K 1978 *Phys. Rev.* D 1901
[13] Biswas S N, Datta K, Saxena R P, Srivastava P K and Varma V S 1971 *Phys. Rev.* D **4** 3617